Sensitivity Analysis of MVL Systems by the Logic Derivatives of MVL Functions

Peter Sedlacek, Maryam Ospanova, Marina Yelis

***Abstract*—**Two approaches of sensitive detection of complex networks realized on multivalued logic (MVL) objects are considered. These approaches are based on algorithms of Logical Differential Calculus and on the ideas of Universal and Directed Partial Logical Derivatives. The distinguishing feature of the proposed algorithms is their computable by regular matrix mathematical apparatus. It allows us to realize the efficient software and hardware support such as homogeneous, parallel and neural structures, and also some standard applied packages of matrix algebra.

***Keywords*—** MVL Function, Logic Differential Calculus, Sensitivity of MVL-systems.

I. Introduction

At present, there are some formal approaches to solve tasks of the sensitivity analysis of logical systems. One of them was based on algorithms of Logical Differential Calculus. The first algorithms are based on the concept of the Logical Derivative of the Boolean function [1] - [3]. Its definition follows from the concept “change” (negation) of binary variable of Boolean function. The development of this approach based on papers [4] and [5]. These algorithms are used for sensitivity analysis of binary systems. However, to solve applied tasks for describing modern logical systems, Multiple Valued Logic (MVL) functions are used instead of Boolean function. Owing to this fact the concept of Boolean Derivative on the area of MVL functions has been generalized [6] - [9]. In this case, the concept of “change” may be defined as inversion or cyclic inversion of logical variables and others. In papers [10] - [12] some types of Logic Derivatives are proposed for sensitivity analysis. These Derivatives depend on the character of the logical connection between the meanings of the variables MVL functions (Derivatives with multi-divisible cyclic inversion of variables, cyclic inversion of variables, negation of variables). The first type of Derivatives detects the modification of the function if the *i*-th variables *xi* sequentially equals to all the values from 0 to *k*-1 (*k*-valued of function). Two other Derivatives allow to detect the character of modification of function if the value corresponding to its cyclic inversion and negation.

The existing algorithms of Logic Different Calculus are based on symbolic transforms of algebra of logic and characterized by large computing complexity and it is difficult to use them in practice. One of the ways to remove this drawback is the realization of these algorithms by matrix apparatus [4], [12] and [13]. Using the matrix mathematical apparatus allows to fulfilling hardware realization on homogeneous, parallel and neural structures or software on standard applied packages of matrix algebra.

In papers [14] - [19] the methods of the detection of changes for multilevel logical systems are designed. However, models of different changes for systems are considered here. Therefore, it needs a complicated combination of several proposed models to sensitivity analysis of these changes. It always requires to apply special software and hardware.

In this paper, two algorithms of Logic Differential Calculus to compute Logic Derivatives of MVL functions are represented. These algorithms allow us to analyse some types of changes.

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II. Main Concepts

*Definition 1.*A k-valued logic (multiple-valued) function $f(X) = f(x_1, x_2, ..., x_n)$ of n variables is the logic function given on the sets $\{0, 1, ..., k\text{-}1\}^n \rightarrow \{0, 1, ..., k\text{-}1\}$[2].

Number of k-valued logic function is equal k^{k^n}. Any k-valued logic function $f(X)$ of n variables may be represented as symbolic or truth-table vector column forms. The symbolic form is

$$f(X) = \bigvee_{l=0}^{k-1} l \wedge P_l(X), \qquad \text{where} \qquad P_l(X) = \bigvee_{f(r_1,r_2,...,r_n)=l} x_1^{r_1} \wedge x_2^{r_2} \wedge \ldots \wedge x_n^{r_n} \qquad (1)$$

what denotes the grouping the terms $x_1^{r_1} \wedge x_2^{r_2} \wedge \ldots \wedge x_n^{r_n}$ on the basis of the same coefficients $l = 0, ..., k\text{-}1$. The literal operation is defined as characteristic function:

$$y^s = \varphi_s(y) = \begin{cases} 0, & y \neq s, \\ k-1, & y = s. \end{cases} \qquad (2)$$

Definition 2. The truth-table column vector is defined as $\mathbf{X} = [x^{(0)}\ x^{(1)} \ldots x^{(k^n-1)}]^T$ given on the set arranged in lexicographic order.

Definition 3. A Partial Logic Derivative of a k-valued logic function $f(X)$ of n variables with respect to variable x_i is defined as follows [12]

$$\partial f(X)/\partial x_i = f(x_1,...,x_i,...,x_n) \oplus f(x_1,..., \overline{x}_i, ...,x_n) \qquad \text{(over GF}(k)), \qquad (3)$$

where the value of variable $\overline{x}_i$ - defines type of changing variable x_i .

Definition 4. A Direct Partial Logic Derivative $\partial f(j\rightarrow l)/\partial x_i(a\rightarrow b)$ of a k-valued logic function $f(X)$ of n variables with respect to variable x_i reflects the fact of changing of function from j to l when the value of variable x_i is changing from a to b and it is computed according to expression [10]:

$$\partial f(j\rightarrow l)/\partial x_i(a\rightarrow b) = P_j(X)\Big|_{x_i=a} \cdot P_l(X)\Big|_{x_i=b}. \qquad (4)$$

Transform this notation (4) in matrix form.

III. Matrix Form of the Universal Partial Logic Derivatives

A matrix form of the Partial Logic Derivative of a k-valued logic function (4) is written by

$$\partial \mathbf{X}/\partial x_i = R_{k^n}^{(i)} \mathbf{X} \qquad \text{(over GF}(k)). \qquad (5)$$

Here $\mathbf{X} = [x^{(0)}\ x^{(1)} \ldots x^{(k^n-1)}]^T$ is the truth-table column vector of the function $f(X)$; the $k^n \times k^n$ differentiation matrix $R_{k^n}^{(i)}$ is formed by the rule

$$R_{k^n}^{(i)} = I_{k^{i-1}} \otimes R_{k^{n-i+1}} \qquad \text{(over GF}(k)), \qquad (6)$$

where $\otimes$ denotes Kronecker product; the elementary matrix $R_{k^{n-i+1}}$ is computed by the rule

$$R_{k^{n-i+1}} = I_{k^{n-i+1}} + (S_k \otimes I_{k^{n-i}}) \qquad \text{(over GF}(k)),$$

where (a, b)-th element of monomial $k \times k$ matrix S_k equals to 1, if $a = x$ and $b = \overline{x}$; $a, b = 0, ..., k\text{-}1$; I_{k^c} - is the $k^c \times k^c$ identity matrix.

Example 1. Compute the differentiation matrix of 3-valued logic function with respect to variable x_1. Let's define the rule of changing variables as the cyclic inversion ($0 \to 1$, $1 \to 2$, and $2 \to 0$). Then matrix S_3 is equal:

$$S_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The differentiation matrix $R^{(1)}_{3^2}$ with respect to variable x_1 is computed by rule (6):

$$R^{(1)}_{3^2} = \left[\begin{array}{ccc|ccc|ccc} 1 & & & 1 & & & & & \\ & 1 & & & 1 & & & & \\ & & 1 & & & 1 & & & \\ \hline & & & 1 & & & 1 & & \\ & & & & 1 & & & 1 & \\ & & & & & 1 & & & 1 \\ \hline 1 & & & & & & 1 & & \\ & 1 & & & & & & 1 & \\ & & 1 & & & & & & 1 \end{array}\right].$$

Example 2. Let's compute the Partial Logic Derivatives of the MVL function $f(X)$ ($n = 2$, $k = 3$) given by its truth-table column vector $\mathbf{X}$=[111 200 210] with respect to variable x_1 using (5):

$$\partial \mathbf{X} / \partial x_1 = R^{(1)}_{3^2} \mathbf{X} = [011\ 110\ 021]^T \qquad \text{(over GF}(k)\text{)},$$

where matrix $R^{(1)}_{3^2}$ was formed in example 1.

The third and the forth elements of the truth-table vector column of the Partial Logical Derivatives $\partial \mathbf{X} / \partial x_1$ equal to 1 ($x^{(3)} = x^{(6)} = 2$). However, in the first case the meaning of the function doesn't depend of the change of variable x_1. In second case the value of the MVL function changes from 0 to 1 when the changing value of c ($x^{(4)} = 0$, $x^{(7)} = 1$). Therefore, these Derivatives cannot be used to synthesis test patterns to test MVL-switching circuits.

Let us consider Universal Logic Derivatives. These Derivatives allow to define the influence of changing the values of input variables on the meaning of the MVL function.

Definition 5. A Universal Partial Logic Derivative of a MVL function with respect to variable x_i is the MVL function defined by

$$\tilde{\partial} f(X) / \tilde{\partial} x_i = \overline{\varphi_s(y)}, \tag{7}$$

where $y = f(x_1, ..., x_i, ..., x_n)$ and $z = f(x_1, ..., \tilde{x}_i, ..., x_n)$.

The matrix form of the Universal Partial Logic Derivatives of MVL function (7) is defined by the expression

$$\tilde{\partial} \mathbf{X} / \tilde{\partial} x_i = R^{(i)}_{k^n} \circ \mathbf{X}, \tag{8}$$

where $R^{(i)}_{k^n}$ - differential matrix formed by the rule (6); symbol $\circ$ denotes comparison operation.

Example 3. Let's compute Universal Partial Logical Derivatives with respect to variable x_1 of MVL function $f(X)$ ($n = 2$, $k = 3$) given by its truth-table column vector $\mathbf{X}$=[111 200 210] using (8):

$$\tilde{\partial} \mathbf{X} / \tilde{\partial} x_1 = R^{(1)}_{3^2} \circ \mathbf{X} = [222\ 020\ 202]^T, \tag{9}$$

where matrix $R^{(1)}_{3^2}$ formed as in Example 1.

The flow graphs of algorithm of computing the Universal Partial Logic Derivatives of MVL-function ($n = 2$, $k = 3$) are shown in Fig.1

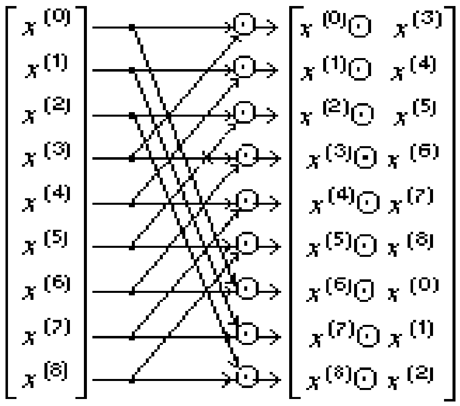

Fig. 1. Flow graphs of the algorithms (9) which respect to variable x_1.

IV. Algorithm of Computing Direct Partial Logical Derivatives

Let the matrix form of the Direct Partial Logic Derivative of MVL functions (3). The truth-table column vector of Direct Partial Logic Derivative of a k-valued logic function f(X) of n variables with respect to variable x_i is computed by the expression [19]:

$$\partial \mathbf{X}(j\rightarrow l)/\partial x_i(a\rightarrow b) = \left(\mathrm{P}_{k^n}^{(i,a)} \cdot \varphi_j(\mathbf{X})\right) \cdot \left(\mathrm{P}_{k^n}^{(i,b)} \cdot \varphi_l(\mathbf{X})\right), \qquad (10)$$

where $\mathbf{X}=[x^{(0)}\ x^{(1)} \dots x^{(k^n-1)}]^{\mathrm{T}}$ is the truth-table column vector of the function f(X), the column vector literal represented as

$$\varphi_s(\mathbf{X}) = \varphi_s([x^{(0)}\ x^{(1)} \dots x^{(k^n-1)}]) = [\ \varphi_s(x^{(0)})\ \varphi_s(x^{(1)}) \dots \varphi_s(x^{(k^n-1)})], \qquad (11)$$

$\varphi_s(y)$ $(s = j, l)$ compute with according of (2).

The matrix $\mathrm{P}_{k^n}^{(i,m)}$ $(m = a, b)$ is formed by the rule

$$\mathrm{P}_{k^n}^{(i,m)} = \mathrm{M}_{k^{i-1}} \otimes \mathrm{P}_k^{(m)} \otimes \mathrm{M}_{k^{n-i}}\ , \qquad (12)$$

where $\mathrm{M}_{k^{i-1}}$ and $\mathrm{M}_{k^{n-i}}$ are the $k^{i-1} \times k^{i-1}$ and $k^{n-I} \times k^{n-i}$ diagonal matrices containing the value (k-1) in main diagonal; the matrix $\mathrm{P}_k^{(m)}$'s structure is the following

$$\mathrm{P}_k^{(m)} = [\ \varphi_m(0)\ \varphi_m(1) \dots \varphi_m(k-1)] \otimes \begin{bmatrix} k-1 & & & \\ & k-1 & & \\ & & \dots & \\ & & & k-1 \end{bmatrix}. \qquad (13)$$

Example 4. Form the matrices $\mathrm{P}_{3^2}^{(1,1)}$ and $\mathrm{P}_{3^2}^{(1,2)}$ using (12,13):

$$\mathrm{P}_{3^2}^{(1,1)} = \mathrm{P}_3^{(1)} \otimes \mathrm{M}_3 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} & & & 2 & & & & & \\ & & & & 2 & & & & \\ & & & & & 2 & & & \\ \hline & & & 2 & & & & & \\ & & & & 2 & & & & \\ & & & & & 2 & & & \\ \hline & & & 2 & & & & & \\ & & & & 2 & & & & \\ & & & & & 2 & & & \end{array}\right]. \quad \mathrm{P}_{3^2}^{(1,2)} = \mathrm{P}_3^{(2)} \otimes \mathrm{M}_3 = \left[\begin{array}{ccc|ccc|ccc} & & & & & & 2 & & \\ & & & & & & & 2 & \\ & & & & & & & & 2 \\ \hline & & & & & & 2 & & \\ & & & & & & & 2 & \\ & & & & & & & & 2 \\ \hline & & & & & & 2 & & \\ & & & & & & & 2 & \\ & & & & & & & & 2 \end{array}\right].$$

Example 5. Compute the Direct Partial Logic Derivative $\partial f(0\rightarrow 1)/\partial x_i(1\rightarrow 2)$ of the 3-valued logic function f(X) of 2 variables given by its truth-table column vector $\mathbf{X}$=[111 200 210]$^{\mathrm{T}}$.

Use expression (10) and write

$$\partial \mathbf{X}(0\rightarrow 1)/\partial x_1(1\rightarrow 2) = \left(P^{(1,1)}_{3^2}\cdot\varphi_0(\mathbf{X})\right)\cdot\left(P^{(1,2)}_{3^2}\cdot\varphi_0(\mathbf{X})\right),$$

where $\varphi_0(\mathbf{X}) = [000\ 022\ 002]^T$, $\varphi_1(\mathbf{X}) = [222\ 000\ 020]^T$; matrices $P^{(1,1)}_{3^2}$ and $P^{(1,2)}_{3^2}$ are formed as in Example 4. The computed column vector $\partial \mathbf{X}(0\rightarrow 1)/\partial x_1(1\rightarrow 2) = [020\ 020\ \ 020]^T$ corresponds to the following symbolic notation: $\partial f(0\rightarrow 1)/\partial x_i(1\rightarrow 2) = x_2^1$.

The flow graphs of algorithm of computing the Direct Partial Logic Derivatives of MVL-function (n = 2, k = 3), a =0, = 1; i = 1, 2 are shown in Fig.2. It should be noted that the structure of synthesized flow graphs coincides with the fast Fourier transform signal flow graphs structure used in digital signal processing.

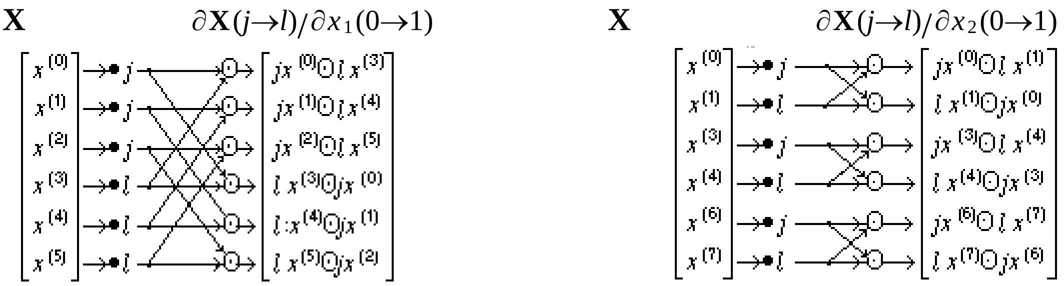

Fig.2. Flow graphs of the algorithms (10), symbol "•" is comparison operation

V. Synthesis of Sets for Sensitivity Analysis

The authors offer simple parallel algorithms for sensitivity analysis of MVL systems. These algorithms are based on the operators of computing of the Universal and Direct Partial Logic Derivatives have been considered in sections III and IV.

Definition 6. Any sets $t_1,..,t_i,...,t_n$ and $t_1,...,\tilde{t}_i,...,t_n$ of input variables $x_1,...,x_i,...,x_n$ detecting the changes for which changing the value of i-th input variable from x_i into $\tilde{x}_i$ leads to changing the value of the realised function from $f(x_1,...,x_i,...,x_n)$ into $f(x_1,...,\tilde{x}_i,...,x_n)$ are called sets for sensitivities analysis.

Let's consider technique of using the Universal and Direct Partial Logic Derivatives to synthesize the sets.

Example 6. We have a logical system realizing the MVL-function f(X) (n = 2, k = 3), defined by its truth-table column vector $\mathbf{X} = [111\ 200\ 210]^T$.

Find the sets x_1x_2 to detect cyclic inversion changes on the logic input x_1.

A) Let's use the Universal Partial Logic Derivatives. Using the results of computing Partial Logical Derivatives received in Example 3 we build Table I.

Table I.

Sets to detect cyclic inversion in input x_1 of the network in Example 6.

Test pattern		Output values	
x_1x_2	$x_1^*x_2$	f(X)	f^*(X)
0 0	1 0	1	2
0 1	1 1	1	0
0 2	1 2	1	0
1 1	2 1	0	1
2 0	0 0	2	1
2 2	0 2	0	1

The sets to be found are sets of variables corresponding to value "2" of the vector $\tilde{\partial}\,\mathbf{X}/\tilde{\partial}\,x_1$.

B) Let's use the Direct Partial Logic Derivatives. To receive the sets it is necessary to compute the Direct Partial Logical Derivatives $\partial \mathbf{X}(j \to k)/\partial x_i(a \to a+1)$ for all j, k, a = 0,1,2 when $a \neq b$. Write the results (non-zero truth vectors of Derivatives) in Table II.

TABLE II

THE NONZERO TRUTH-TABLE COLUMN VECTORS OF THE PARTIAL LOGIC DERIVATIVES

$\frac{\partial X(1\to 2)}{\partial x_1(0\to 1)}$	$\frac{\partial X(1\to 0)}{\partial x_1(0\to 1)}$	$\frac{\partial X(1\to 2)}{\partial x_1(0\to 2)}$	$\frac{\partial X(0\to 1)}{\partial x_1(1\to 2)}$	$\frac{\partial X(1\to 0)}{\partial x_1(0\to 2)}$
2	0	2	0	0
0	2	0	2	0
0	2	0	0	2
2	0	2	0	0
0	2	0	2	0
0	2	0	0	2
2	0	2	0	0
0	2	0	2	0
0	2	0	0	2

Nonzero elements of the truth-table column vectors of the Derivatives define the variables sets being the sets to detect the cyclic inversion faults in the given circuit (Table III). Thus the sets (Table I) allow to detect cyclic inversion changes in input x_1.

TABLE III

CHANGE OF VALUES OF VARIABLES AND FUNCTION

Change of variable x_1 value	Change of function' value		
	$\partial\mathbf{X}(0\to1))$	$\partial\mathbf{X}(1\to0)$	$\partial\mathbf{X}(1\to2)$
$\partial x_1(0\to1)$		x_2 =1 x_2 =2	x_2 =0
$\partial x_1(0\to2)$		x_2 =2	x_2 =0
$\partial x_1(1\to2)$	x_2 =1		

Both the suggested technologies have the same computational complexity. Its distinguish features detect on the stage of hardware and software realisation. One of the main characteristics of the received technologies is the possibility of their realisation following with different hardware architecture principles (parallel, pipelining, systolic etc.) [12] and software [20]. The initial object should be described by truth-table vector columns. As an example, these descriptions can be obtained based on the induction of Fuzzy Decision Trees (FDT). The process of calculation of truth-table vector columns based on FDT is presented in papers [21] and [22].

CONCLUSION

The main approaches suggested here on sensitivity analysis of MVL systems are based on different mathematical concepts of Logic Derivatives for MVL-functions. It allows to fulfil a flexible adaptation of theoretical results to solve applied diagnostics tasks that have a principal meaning for software and hardware realisation of these algorithms.

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